LINES INDUCED BY BICHROMATIC POINT SETS

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Abstract. An important theorem of Beck says that any point set in the Euclidean plane is either "nearly general position" or "nearly collinear": there is a constant $C > 0$ such that, given n points in $\mathbb{E}^2$ with at most r of them collinear, the number of lines induced by the points is at least $Cr(n-r)$.

Recent work of Gutkin-Rams on billiards orbits requires the following elaboration of Beck's Theorem to bichromatic point sets: there is a constant $C > 0$ such that, given n red points and n blue points in $\mathbb{E}^2$ with at most r of them collinear, the number of lines spanning at least one point of each color is at least $Cr(2n-r)$.

1. Introduction

Let $\mathbf{p}$ be a set of n points in the Euclidean plane $\mathbb{E}^2$ and let $\mathcal{L}(\mathbf{p})$ be the set of lines induced by $\mathbf{p}$. A line $\ell \in \mathcal{L}(\mathbf{p})$ is k-rich if it is incident on at least k points of $\mathbf{p}$. A well-known theorem of Beck relates the size of $\mathcal{L}(\mathbf{p})$ and the maximum richness.

Theorem 1 (**Beck's Induced Lines Theorem [1]**). *Let $\mathbf{p}$ be a set of n points in $\mathbb{E}^2$, and let r be the maximum richness of any line in $\mathcal{L}(\mathbf{p})$. Then $\left|\mathcal{L}(\mathbf{p})\right| \gg r(n-r)$.*

Here $f(n) \gg g(n)$ means that $f(n) \geq Cg(n)$, for an absolute constant $C > 0$.

In this note, we give an elaboration (using pretty much the same arguments) of Beck's Theorem to bichromatic point sets, which arises in relation to the work of Gutkin and Rams [2] on the dynamics of billiard orbits. Let $\mathbf{p}$ be a set of n red points and let $\mathbf{q}$ be a set of n blue points with all points distinct (for a total of $2n$). We define $\mathbf{p} \cup \mathbf{q}$ to be the *bichromatic point set* $(\mathbf{p}, \mathbf{q})$ and define the set of *bichromatic induced lines* $\mathcal{B}(\mathbf{p}, \mathbf{q})$ to be the subset of $\mathcal{L}(\mathbf{p}, \mathbf{q})$ that is incident on at least one point of each color.

Theorem 2 (**Beck-type theorem for bichromatic point sets**). *Let $(\mathbf{p}, \mathbf{q})$ be a bichromatic point set with n points in each color class (for a total of $2n$). If the maximum richness of any line in $\mathcal{L}(\mathbf{p}, \mathbf{q})$ is r, then $\left|\mathcal{B}(\mathbf{p}, \mathbf{q})\right| \gg n(2n-r)$.*

In the particular case where $r = n$, which is required by Gutkin and Rams, this shows that $\left|\mathcal{B}(\mathbf{p}, \mathbf{q})\right| \gg n^2$.

Beck's Theorem 1, and the present Theorem 2, may be deduced from the famous Szemeredi-Trotter Theorem on point-line incidences (Beck himself uses a weaker, but similar, statement as his key lemma). The following form is what we require in the sequel.

Theorem 3 (**Szemeredi-Trotter Theorem [4]**). *Let $\mathbf{p}$ be a set of n points in $\mathbb{E}^2$ and let $\mathcal{L}$ be a finite set of lines in $\mathbb{E}^2$. Then the number r of k-rich lines in $\mathcal{L}$ is $r \ll n^2/k^3 + n/k$.*

Notations. We use $\mathbf{p} = (\mathbf{p}_i)_1^n$ and $\mathbf{q} = (\mathbf{q}_i)_1^n$ for point sets in $\mathbb{E}^2$. The notation $f(n) \gg Cg(n)$ means there is an absolute constant $C > 0$ such that $f(n) \geq g(n)$ for all $n \in \mathbb{N}$.

2. Proofs

The proof of the main theorem follows a similar line to Beck's original proof. For a pair of points $(\mathbf{p}_i, \mathbf{q}_j)$, we define the richness of the pair to be the richness of the line $\mathbf{p}_i\mathbf{q}_j$.

Lemma 4. *Let $(\mathbf{p}, \mathbf{q})$ be a bichromatic point set in $\mathbb{E}^2$. Then there is an absolute constant $K_1 > 0$ such that the number of bichromatic point pairs that are either at most $1/K_1$-rich or at least $K_1 n$-rich is at least $n^2/2$.*

Proof. There are exactly n^2 pairs of points $(\mathbf{p}_i, \mathbf{q}_j)$, and each of these induces a line in $\mathcal{B}(\mathbf{p}, \mathbf{q})$. Define the subset $\mathcal{B}_j(\mathbf{p}, \mathbf{q}) \subset \mathcal{B}(\mathbf{p}, \mathbf{q})$ to be the set of bichromatic lines with richness between 2^{j-1} and 2^j.

By the Szemeredi-Trotter Theorem with $k = 2^j$,

$$\left|\mathcal{B}_j(\mathbf{p}, \mathbf{q})\right| \leq C(n^2/2^{3j} + n/2^j) \tag{1}$$

The number of bichromatic pairs inducing any line $\ell \in \mathcal{B}_j(\mathbf{p}, \mathbf{q})$ is maximized when there are 2^j red points and 2^j blue ones on ℓ, for a total of 2^{2j} bichromatic pairs. Multiplying by the estimate of (1), the number of bichromatic pairs inducing lines in $\mathcal{B}_j(\mathbf{p}, \mathbf{q})$ is at most

$$C(n^2/2^j + n2^j) \tag{2}$$

for a large absolute constant C coming from the Szemeredi-Trotter Theorem.

Now let $K_1 > 0$ be a small constant to be selected later. We sum (2) over j such that $1/K_1 \leq j \leq K_1 n$:

$$C\left(n^2 \sum_{1/K_1 \leq j \leq K_1 n} 2^{-j} + n \sum_{1/K_1 \leq j \leq K_1 n} 2^j\right) \leq C\left(n^2 2^{-1/K_1} + n \cdot 2K_1 n\right)$$

Picking K_1 small enough (it depends on C) ensures that at most $n^2/2$ of the monochromatic pairs induce lines with richness between $1/K_1$ and $K_1 n$. □

The following lemma is a bichromatic variant of Beck's Two-Extremes Theorem.

Lemma 5. *Let $(\mathbf{p}, \mathbf{q})$ be a bichromatic point set in $\mathbb{E}^2$. Then either*

- **A** *The number of bichromatic lines $\left|\mathcal{B}(\mathbf{p}, \mathbf{q})\right| \gg n^2$.*
- **B** *There is a line $\ell \in \mathcal{B}(\mathbf{p}, \mathbf{q})$ incident on at least $K_2 n$ red points and $K_2 n$ blue points for an absolute constant $K_2 > 0$.*

Proof. We partition the bichromatic pairs $(\mathbf{p}_i, \mathbf{q}_j)$ into three sets: L is the set of pairs with richness less than $1/K_1$; M is the set of pairs with richness in the interval $[1/K_1, K_1 n]$; H is the set of pairs with richness greater than $K_1 n$.

By Lemma 4, $|L \cup H| \geq n^2/2$. There are now three cases:

Case I: (Alternative A) If $|L| \geq n^2/4$, then we are in alternative **A**, since quadratically many pairs can be covered only by quadratically many lines of constant richness.

Case II: (Alternative B) If we are not in Case I, then, $|H| \geq n^2/4$. In particular, since H is not empty, there are lines incident on at least one point of each color and at least $K_1 n$ points in total. If one of these lines is line incident to at least $\frac{K_1}{6} n$ points of each color, then we are in alternative **B**.

Case III: (Alternative A) If we are not in Case I or Case II, then every line induced by a bichromatic pair in H has at least $\frac{5}{6} K_1 n$ red points incident on it or $\frac{5}{6} K_1 n$ blue ones incident on it.

Since there are $|H| \geq n^2/4$ bichromatic point pairs incident on a very rich line, there must be at least $n/4$ different points of each color participating in some point pair in H.

Each line induced by a pair in H generates at least $K_1 n$ incidences, so the number of these lines is at most $\frac{1}{K_1}n$. But then if all the lines induced by H span at most $\frac{1}{6}K_1 n$ blue points, the total number of blue incidences is less than $n/4$, which is a contradiction. We can make a similar argument for red points.

Thus there is a line ℓ_1 spanning at least $\frac{5}{6}K_1 n$ blue points and a distinct line ℓ_2 spanning at least $\frac{5}{6}K_1 n$ red points. From this configuration we get at least $(\frac{5}{6}K_1 n - 1)^2$ distinct bichromatic lines, putting us again in alternative **A**. □

Proof of Theorem 2. If alternative **A** of Lemma 5 holds, then we are already done.

If we are in alternative **B**, then there must be a line ℓ of richness $r \geq 2K_2 n$ incident to at least $K_2 n$ points of each color. Now pick any subset X of $K_2(2n - r)$ points not incident to ℓ. There are at least $\frac{1}{2}K_2^2 n(2n - r)$ bichromatic point pairs determined by one point in X and one point incident to ℓ. Thus we get at least

$$\frac{1}{2}K_2^2 n(2n - r) - \binom{K_2(2n - r)}{2} \geq \frac{1}{2}K_2^3 n(2n - r)$$

bichromatic lines. □

3. Conclusion

We proved an extension of Beck's Theorem [1] to bichromatic point sets using a fairly standard argument, completing the combinatorial step in Gutkin-Rams's recent paper on billiards.

This kind of bichromatic result can, due to the general nature of the proofs, be extended to any setting where a Szemeredi-Trotter-type result is available (see, e.g., [3] for many examples). Moreover, by "forgetting" colors and repeatedly squaring the constants, Theorem 2 holds for bichromatic lines in multi-chromatic point sets. It would, however, be interesting to know whether this is the correct order of growth for the constants in a multi-chromatic version of Theorem 2.

References

[1] József Beck. On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry. *Combinatorica*, 3(3-4):281–297, 1983. ISSN 0209-9683. doi: 10.1007/BF02579184.

[2] Eugene Gutkin and Michael Rams. Complexities for cartesian products of homogeneous Lagrangian systems, with applications to billiards. Preprint, 2010.

[3] László A. Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.*, 6(3):353–358, 1997. ISSN 0963-5483. doi: 10.1017/S0963548397002976.

[4] Endre Szemerédi and William T. Trotter, Jr. Extremal problems in discrete geometry. *Combinatorica*, 3(3-4):381–392, 1983. ISSN 0209-9683. doi: 10.1007/BF02579194.

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